p-adic Galois representations and (ϕ, Γ)-modules

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Abstract

In this note we present a basic theory of p-adic Galois representations and (ϕ, Γ)-modules. In particular, we prove a series of equivalences between both (1-)categories over various rings following Fontaine and Cherbonnier–Colmez.

Contents

1 Notations

Suppose K is a field.

Let G_K denote the absolute Galois group of K.

Let $\chi\colon G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be the cyclotomic character.

If K is a finite extension of $\mathbb{Q}_p$, then let K_∞ be the infinite cyclotomic extension over K. Let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ in K_∞ and k_{K_∞} be the residue field of K_∞. Let $H_K := \ker(\chi|_{G_K}) \cong G_{K_\infty}$ and $\Gamma_K := G_K/H_K \cong \mathrm{Gal}(K_\infty/K)$ by local class field theory.

For a commutative ring R, let $\mathbb{W}_P(R)$ denote the ring of p-typical Witt vectors over R.

Let R be a topological ring and G be a topological group acting continuously on R. Let $\mathrm{Rep}_R(G)$ denote the abelian category of continuous (finite free) R-representations of G.

2 Perfectoid rings and tiltings

The idea of the perfectoid rings is to show a correspondence between local fields of mixed characteristic and equal characteristic. In this section, we give a brief introduction to the basic settings in perfectoid rings, basically following [SW20].

The content of this section will not be heavily used in the following sections. We include them here because it provides a modern approach to Corollary 2.20 and for future study in p-adic geometry beyond this note.

2.1 Huber rings

Definition 2.1 (Huber ring)**.** A topological ring A is ***Huber*** if A admits an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ such that $\{I^n\colon n \geqslant 0\}$ forms a basis of neighborhoods of 0.

Any such A_0 is called ***a ring of definition of*** A.

Example 2.2.

1. *$(\mathbb{Q}_p, \mathbb{Z}_p)$ and $(\mathbb{Q}_p, \mathbb{Q}_p)$ are both Huber.*

2. *If k is a perfect field of characteristic p, then $(\mathbb{W}_P(k)[\![x_1, \cdots, x_n]\!], \mathbb{W}_P(k)[\![x_1, \cdots, x_n]\!])$ is Huber with respect to the $(p, x_1, \cdots, x_n)$-adic topology. This ring classifies deformations of formal group laws and shows up further in chromatic homotopy theory.*

There is a simple characterization of a ring of definition via boundedness.

Definition 2.3 (Bounded subset)**.** A subset S of a topological ring A is ***bounded*** if for all open neighborhoods U of 0, there exists an open neighborhood V of 0 such that $VS \subset U$.

Lemma 2.4 (cf. [SW20, Lemma 2.2.4])**.** *A subring A_0 of a Huber ring A is a ring of definition if and only if A_0 is open and bounded.*

The universal ring of definition is given by the so-called power-bounded elements.

Definition 2.5 (Power-bounded elements)**.** Let A be a Huber ring. An element $x \in A$ is ***power-bounded*** if the subset $\{x^n : n \geqslant 0\}$ is bounded. Let $A^\circ \subset A$ be the subring of power-bounded elements.

Proposition 2.6.

1. *Any ring of definition $A_0 \subset A$ is contained in A°.*
2. *The ring A° is the filtered union of the rings of definition $A_0 \subset A$.*

Proof.

1. Suppose $x \in A_0$, so $x^n \in A_0$ for any $n \geqslant 0$. Since A_0 is bounded by the above lemma, $x \in A^\circ$.
2. We first show that the poset of rings of definition is filtered. Suppose $A_0, A_0' \subset A$ are rings of definition. Let $A_0'' \subset A$ be the subring generated by A_0, A_0'. For any $U \subset A$ open neighborhood of 0, we want to find an open neighborhood $V \subset A$ of 0 such that $VA_0'' \subset U$. We may assume that U is closed under addition (in fact, we can take $U = I^n$, where I is the ideal in the definition of A and A_0). Then there is an open neighborhood $U_1 \subset A$ of 0 such that $U_1 A_0 \subset U$ and there is an open neighborhood $V \subset A$ of 0 such that $VA_0' \subset U_1$. Any element in A_0'' can be written as a linear combination $\sum_i x_i y_i$ where $x_i \in A_0$ and $y_i \in A_0'$. Thus, we have

$$(\sum_i x_i y_i)V \subset \sum_i (x_i y_i V) \subset \sum_i x_i U_1 \subset \sum_i U \subset U$$

   Therefore, A_0'' is bounded and further a ring of definition by the above lemma.

   Now pick an element $x \in A^\circ$. Suppose A_0 is a ring of definition. Then $A_0[x]$ is still a ring of definition since it is still bounded.

□

Definition 2.7 (Uniform Huber ring)**.** A Huber ring A is ***uniform*** if A° is bounded, or equivalently, A° is a ring of definition.

Definition 2.8 (Huber pair and ring of integral elements)**.** A ***Huber pair*** is a pair (A, A^+), where A is a Huber ring and $A^+ \subset A$ is an open and integrally closed subring of A.

Such A^+ is called ***a ring of integral elements***.

Let $A^{\circ\circ} \subset A$ be the subset of topologically nilpotent elements. For any $x \in A^{\circ\circ}$, $x^n \in A^+$ for n large enough since A^+ is open. Therefore, x must lie in A^+ since A^+ is integrally closed, so we have $A^{\circ\circ} \subset A^+$ for any ring of integral elements A^+.

To sum up, we have the following inclusions between subrings in a Huber ring A.

$$\begin{array}{ccccccc} A^{\circ\circ} & \hookrightarrow & A^+ & \hookrightarrow & A^\circ & \hookrightarrow & A \\ & & & & \uparrow\wr & & \\ & & A_0 & \hookrightarrow & \bigcup A_0 & & \end{array}$$

where the union is filtered and is taken over all rings of definition A_0 in A.

2.2 Perfectoid rings

Definition 2.9 (Tate ring and pseudo-uniformizer)**.** A Huber ring A is ***Tate*** if it contains a topologically nilpotent unit. A ***pseudo-uniformizer*** in A is a topologically nilpotent unit.

Definition 2.10 (Perfectoid ring and perfectoid field)**.** A complete Tate ring R is ***perfectoid*** if R is uniform and there exists a pseudo-uniformizer ϖ of R lives in R° such that p divides ϖ^p in R°, and the p-th power Frobenius map

$$\phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$$

is an isomorphism.

A ***perfectoid field*** is a perfectoid ring R which is a non-archimedean field.

Proposition 2.11. *Suppose R is a complete Tate ring that admits a pseudo-uniformizer ϖ of R lives in R° such that p divides ϖ^p in R°. Then the p-th power Frobenius map $\phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism if and only if the Frobenius map $R^\circ/p \to R^\circ/p$ is surjective.*

In particular, the above definition does not depend on the choice of ϖ.

Proof. If $x \in R^\circ$ and $x^p = \varpi^p y$ for some $y \in R^\circ$, then $(x/\varpi)^p \in R^\circ$. By the definition of R°, $x/\varpi \in R^\circ$. Therefore, ϕ is always injective.

We have a commutative diagram.

$$\begin{array}{ccccc} R^\circ & \twoheadrightarrow & R^\circ/p & \longrightarrow & R^\circ/p \\ & \searrow & & & \downarrow \\ & & R^\circ/\varpi & \xrightarrow{\phi} & R^\circ/\varpi^p \end{array}$$

Thus, the surjectivity of the Frobenius on R°/p implies the surjectivity of ϕ.

Conversely, if ϕ is surjective, then for any $x \in R^\circ$, we can approximate x successively via ϕ since ϖ is topologically nilpotent and R is complete, i.e., $x = x_0^p + x_1^p\varpi^p + x_2^p\varpi^{2p} + \cdots$ for some $x_0, x_1, \cdots \in R^\circ$. Thus, $x - (x_0 + x_1\varpi + x_2\varpi^2 + \cdots) \in pR^\circ$. □

Proposition 2.12 (cf. [SW20, Proposition 6.1.6])**.** *Let R be a complete Tate ring of characteristic p. Then R is perfectoid if and only if R is perfect.*

Proposition 2.13 (cf. [SW20, Proposition 6.1.9])**.** *Let K be a non-archimedean field. Then K is a perfectoid field if and only if the following conditions hold.*

1. *K is not discretely valued.*

2. *$|p| < 1$.*

3. *$\phi\colon \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.*

We give the following examples of perfectoid rings without proof. Some of them can be found in [SW20, Example 6.1.5].

Example 2.14.

1. *If A is perfectoid, A° is also perfectoid.*

2. *By the above criterion, $\mathbb{Q}_p$ is not perfectoid, nor any finite extension of $\mathbb{Q}_p$.*

3. *The p-adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}_p}$ is perfectoid.*

4. *The p-adic completion $\mathbb{Q}_p^{\mathrm{cycl}}$ of $\mathbb{Q}_p(\mu_{p^\infty})$ is perfectoid.*

5. *The integer rings of $\mathbb{C}_p$ and $\mathbb{Q}_p^{\mathrm{cycl}}$ are also perfectoid.*

6. *Suppose K is a finite extension of $\mathbb{Q}_p$. Fix a uniformizer π of K and a Lubin-Tate formal group law $F \in \mathcal{O}_K[\![X, Y]\!]$. Then the p-adic completion of K_π associated to F in explicit local class field theory by Lubin and Tate is a perfectoid field.*

7. *The T-adic completion $\mathbb{F}_p(\!(T^{1/p^\infty})\!)$ of $\mathbb{F}_p(\!(T)\!)(T^{1/p^\infty})$ is perfectoid.*

2.3 Tilting and the equivalence of étale sites

Definition 2.15 (Tilt)**.** Let R be a perfectoid ring. The ***tilt*** of R is

$$R^\flat := \lim_{x\mapsto x^p} R$$

with the limit topology. A priori this is only a topological multiplicative monoid. In particular, we have a continuous and multiplicative map $(-)^\sharp\colon R^\flat \to R$ projecting to the first coordinate. Furthermore, we can promote $R^\flat$ to a topological ring where the addition is given by

$$(x_0, x_1, \cdots) + (y_0, y_1, \cdots) := (z_0, z_1, \cdots)$$

where

$$z_i := \lim_{n\to+\infty} (x_{i+n} + y_{i+n})^{p^n}.$$

Note that $(-)^\sharp$ is not additive.

Lemma 2.16 (cf. [SW20, Lemma 6.2.2])**.** *1. The above addition promotes $R^\flat$ to a topological perfect F_p-algebra.*

2.

$$R^{\flat\circ} \cong R^{\circ\flat} := \lim_{x\mapsto x^p} R^\circ \cong \lim_{x\mapsto x^p} R^\circ/p \cong \lim_{\phi} R^\circ/\varpi$$

where $\varpi \in R^\circ$ is a pseudo-uniformizer which divides p in R°.

3. There exists a pseudo-uniformizer ϖ of R lives in R° such that p divides ϖ^p in R°, and admits a sequence of p-th power roots ϖ^{1/p^n} in R°, and the sequence $\varpi^\flat := (\varpi, \varpi^{1/p}, \cdots) \in R^{\flat\circ}$ is a pseudo-uniformizer of $R^\flat$. Furthermore, $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$.

Remark 2.17. *Suppose K is a perfectoid field. Then the composition $K^\flat \xrightarrow{(-)^\sharp} K \xrightarrow{|\cdot|} \mathbb{R}_{\geqslant 0}$ promotes $K^\flat$ to a complete non-archimedean field of characteristic p.*

Example 2.18 (cf. [SW20, Example 6.2.4])**.** *Let $\zeta_p, \zeta_{p^2}, \cdots$ be a compatible system of p-th power roots of unity in $\mathbb{Q}_p^{\mathrm{cycl}}$, $\epsilon := (1, \zeta_p, \zeta_{p^2}, \cdots) \in (\mathbb{Q}_p^{\mathrm{cycl}})^\flat$. Then $\bar{\pi} := \epsilon - 1$ is a pseudo-uniformizer of $(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$. In fact, $(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \cong \mathbb{F}_p(\!(T^{1/p^\infty})\!)$ sending $\bar{\pi}$ to T.*

Theorem 2.19 (The equivalence of étale sites, cf. [SW20, Theorem 7.3.1 and Theorem 7.3.2])**.** *Let K be a perfectoid field. Then there is an equivalence between the sites of finite étale algebras over K and over $K^\flat$.*

Corollary 2.20. *We have that* $G_{(\mathbb{Q}_p^{\mathrm{cycl}})^\flat} \cong G_{\mathbb{Q}_p^{\mathrm{cycl}}} \cong H_{\mathbb{Q}_p}$.

Thus, instead of working over $\mathbb{Q}_p^{\mathrm{cycl}}$, we can move to its tilt, which is of characteristic p.

3 Travel through a series of rings

Now we will define a series of rings in p-adic Galois representations. The goal is to transfer from the original base rings of p-adic Galois representations, such as $\mathbb{F}_p, \mathbb{Z}_p$ and $\mathbb{Q}_p$, to rings that carry more structures while preserve the Galois groups.

Various but similar notations of rings are very confusing for a first read. It is always a good idea to keep in mind a picture of ring extensions. The rules of naming the rings are the following.

The letter A stands for a topological ring with a non-archimedean valuation, B stands for inverting p in A (most time B stands for a field and A will stand for the integer ring of B), and E stands for the reduction of A modulo p. The rings with tilde will always be larger than the one without tilde.

3.1 Rings of characteristic p

We will start with the series of rings named by E, which will deal with the p-adic Galois representations over $\mathbb{F}_p$.

Let $\tilde{E} := \mathbb{C}_p^\flat$, $\tilde{E}_{\mathbb{Q}_p} := (\mathbb{Q}_p^{\mathrm{cycl}})^\flat$ and $E_{\mathbb{Q}_p} := \mathbb{F}_p(\!(T)\!)$. Let $\epsilon := (1, \zeta_p, \zeta_{p^2} \cdots)$ for a chosen compatible system of p-th power roots of unity and $\bar{\pi} := \epsilon - 1$ as in Example 2.18. Define the non-archimedean valuation val_E on $\tilde{E}$ via Remark 2.17. Then

$$\mathrm{val}_{\tilde{E}}(\bar{\pi}) = \mathrm{val}_p(\lim_{n\to+\infty}(\zeta_{p^n} - 1)^{p^n}) = \lim_{n\to+\infty} p^n \mathrm{val}_p(\zeta_{p^n} - 1) = \frac{p}{p-1} > 0.$$

Thus, there is an inclusion $E_{\mathbb{Q}_p} \hookrightarrow \tilde{E}_{\mathbb{Q}_p}$ given by $T \mapsto \bar{\pi}$. Let $E := \mathbb{F}_p(\!(T)\!)^{\mathrm{sep}}$ in $\tilde{E}$. In other words, we have the following diagram of field extensions.

$$\begin{array}{ccc}
\mathbb{C}_p^\flat =: \tilde{E} & \hookleftarrow & E := \mathbb{F}_p(\!(T)\!)^{\mathrm{sep}} \\
| & & | \\
(\mathbb{Q}_p^{\mathrm{cycl}})^\flat =: \tilde{E}_{\mathbb{Q}_p} & \hookleftarrow & E_{\mathbb{Q}_p} := \mathbb{F}_p(\!(T)\!)
\end{array}$$

All of these rings are characteristic p. Thus, they carry an action by the Frobenius map ϕ. Note that $\tilde{E}$ and $\tilde{E}_{\mathbb{Q}_p}$ are perfect while E and $E_{\mathbb{Q}_p}$ are not. Furthermore, $\tilde{E} := \mathbb{C}_p^\flat$ carries an action by $G_{\mathbb{Q}_p}$ component-wise.

Theorem 3.1 (cf. [Ber10, Theorem 15.4])**.** *The canonical map $H_{\mathbb{Q}_p} \cong G_{\tilde{E}_{\mathbb{Q}_p}} \to \mathrm{Gal}(E/E_{\mathbb{Q}_p})$ is an isomorphism.*

Recall that the first isomorphism here is given by Corollary 2.20.

If K is a finite extension of $\mathbb{Q}_p$, let $E_K := E^{H_K}$, which is a finite extension of $E_{\mathbb{Q}_p}$ by the above theorem and Galois correspondence.

Lemma 3.2. *If $\bar{\pi}_K$ is a uniformizer of E_K, then $T \mapsto \bar{\pi}_K$ defines an isomorphism $k_{K_\infty}(\!(T)\!) \cong E_K$.*

Proof. Since E_K is a finite extension of $E_{\mathbb{Q}_p} := \mathbb{F}_p(\!(T)\!)$ and the residue field of E_K is k_{K_∞}, we conclude by the structure theorem for local fields of equal characteristic. □

We have the following generalization of Hilbert's Theorem 90 and its corollary.

Proposition 3.3 (cf. [Ber10, Corollary 7.3])**.** *Let L/K be a Galois extension with $G := \mathrm{Gal}(L/K)$. If we equip L with the discrete topology, then $H^1_{\mathrm{cts}}(G, L) = 0$ and $H^1_{\mathrm{cts}}(G, \mathrm{GL}_n(L)) = 0$ for all $n \geqslant 1$.*

Theorem 3.4. *If K is a finite extension of $\mathbb{Q}_p$, then $H^1_{\mathrm{cts}}(H_K, E) = 0$ and $H^1_{\mathrm{cts}}(H_K, \mathrm{GL}_n(E)) = 0$ for all $n \geqslant 1$. Here E is equipped with the discrete topology.*

3.2 Rings of characteristic 0

Next, we will introduce the series of rings named by A and B, which will deal with p-adic Galois representations over $\mathbb{Z}_p$ and $\mathbb{Q}_p$ respectively.

A canonical way to transfer from characteristic p to characteristic 0 is via the p-typical ring of Witt vectors. However, since E is not perfect, $\mathbb{W}_P(E)$ is not well-behaved (say, $\mathbb{W}_P(E)$ is not p-adically complete and elements in $\mathbb{W}_P(E)$ do not have a series representation). Thus, we need more works to lift the rings E and $E_{\mathbb{Q}_p}$ to characteristic 0.

Firstly, we want to introduce the weak topology on the ring of p-typical Witt vectors. Suppose R is a perfect ring of characteristic 0 complete with respect to a valuation val. Then for each $k > 0$, define $w_k\colon \mathbb{W}_P(R) \to \mathbb{R} \cup \{+\infty\}$ by $w_k(x) := \inf_{i \leqslant k} \mathrm{val}(x_i)$ for

$x = \sum_{i>0} p^i [x_i]_P$ in $\mathbb{W}_P(R)$, where $[-]_P$ is the Teichmüller representative. Then $w_k(x) = +\infty$ if and only if $x \in p^{k+1}\mathbb{W}_P(R)$ and $w_k(x+y) \geqslant \inf(w_k(x), w_k(y))$ for all $x, y \in \mathbb{W}_P(R)$.

Definition 3.5 (Weak topology on the ring of p-typical Witt vectors)**.** The ***weak topology on*** $\mathbb{W}_P(R)$ is the topology defined by w_k for all k.

Proposition 3.6 (cf. [Ber10, Proposition 16.4])**.** *The ring $\mathbb{W}_P(R)$ is complete with respect to the weak topology.*

Let $\tilde{A} := \mathbb{W}_P(\tilde{E}) := \mathbb{W}_P(\mathbb{C}_p^\flat)$ and $\tilde{B} := \tilde{A}[1/p] := \mathbb{W}_P(\mathbb{C}_p^\flat)[1/p]$. Note that $\tilde{A}$ is equipped with both p-adic topology and the weak topology discussed above. Furthermore, $\tilde{A}$ is complete with respect to both topologies. We equip $\tilde{B} = \cup_{k>0} p^{-k}\tilde{A}$ with the colimit topologies of the p-adic topology and the weak topology on $\tilde{A}$ respectively. As a ring of Witt vectors, $\tilde{A}$ is equipped with a Frobenius map ϕ. Also, there is a lift of the $G_{\mathbb{Q}_p}$-action on $\tilde{E}$ to $\tilde{A}$.

In order to lift the rings E and E_K to characteristic 0, where K is a finite extension of $\mathbb{Q}_p$, let $A_K := (\mathbb{W}_P(k_{K_\infty})((T)))_p^\wedge$ and $B_K := A_K[1/p]$. Similar to the inclusions $E_{\mathbb{Q}_p} \hookrightarrow \tilde{E}_{\mathbb{Q}_p} \hookrightarrow \tilde{E}$, we have an inclusion $A_K \hookrightarrow \tilde{A}$ given by $T \mapsto [\bar{\pi}_K]_P$, where $\bar{\pi}_K$ is a uniformizer of $E_K \cong k_{K_\infty}((T)) \hookrightarrow \tilde{E}$. Note that $w_k([\bar{\pi}_K]) = \mathrm{val}_E(\bar{\pi}_K) > 0$, so this map is well-defined. This map also extends to an inclusion $B_K \hookrightarrow \tilde{B}$.

Let $A := (\mathrm{colim}_K A_K)_p^\wedge$ and $B := A[1/p]$, where K runs through all finite extensions of $\mathbb{Q}_p$. Then $A/pA \cong \mathrm{colim}_K E_K = E$. By construction, A_K inherits a $G_{\mathbb{Q}_p}$-action from $\tilde{A}$ and is fixed by H_K.

Lemma 3.7 (cf. [BC09, Lemma 13.5.7])**.** *For each finite extension K of $\mathbb{Q}_p$, $A^{H_K} = A_K$.*

Since we mapped T to $[\bar{\pi}_K]_P$ in the construction of A_K, A_K is ϕ-stable in $\tilde{A}$. Thus, A and B are ϕ-stable in $\tilde{A}$.

The following proposition is a corollary of Proposition 3.3.

Proposition 3.8 (cf. [Ber10, Proposition 7.4])**.** *Let ϖ be a topologically nilpotent element of a ring R which is complete for the ϖ-adic topology and in which ϖ is not a zero divisor. Let G be a group which acts on R continuously and fixes ϖ.*

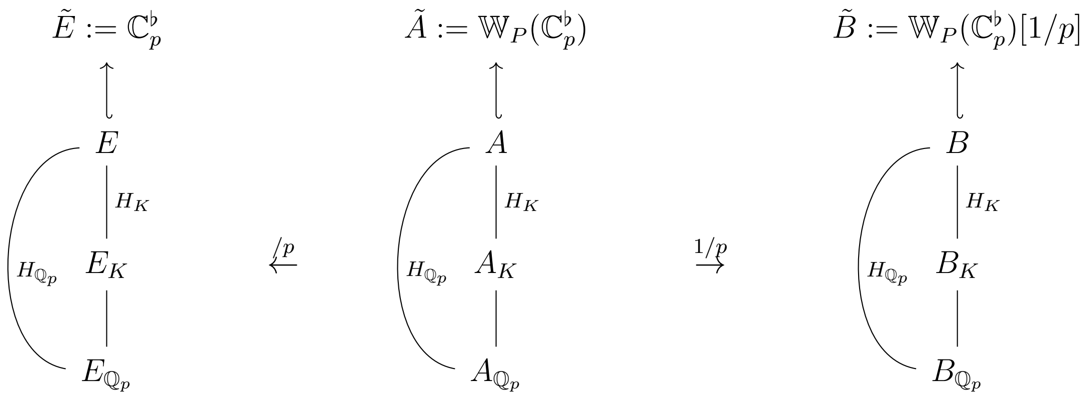
If $H^1_{\mathrm{cts}}(G, \mathrm{GL}_n(R/\varpi R)) = H^1_{\mathrm{cts}}(G, R/\varpi R) = 0$ and the map $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\varpi R)$ is surjective, then $H^1_{\mathrm{cts}}(G, \mathrm{GL}_n(R)) = H^1_{\mathrm{cts}}(G, R) = 0$.

Theorem 3.9. *If K is a finite extension of $\mathbb{Q}_p$, then $H^1_{\mathrm{cts}}(H_K, A) = 0$ and $H^1_{\mathrm{cts}}(H_K, \mathrm{GL}_n(A)) = 0$ for all $n \geqslant 1$. Here A is equipped with the p-adic topology.*

Proof. Note that p is a topologically nilpotent element of A, A is p-complete and p is not a zero-divisor in A. Every lift of $\mathrm{GL}_n(E)$ to $\mathrm{Mat}_n(A)$ has determinant not in pA. Since $A/pA \cong E$ is a field, every lift is invertible. Therefore, we conclude by the above proposition and Theorem 3.4. □

Note that B/B_K is not algebraic, so we cannot prove $H^1_{\mathrm{cts}}(H_K, B) = H^1_{\mathrm{cts}}(H_K, \mathrm{GL}_d(B)) = 0$ via Proposition 3.3. Besides, we cannot use the above proposition since B is a field.

To sum up, we have the following diagram of extensions of rings.



where the rings without tilde named by E, A and B are Laurent series, p-adic completion of Laurent series and p-adic completion of Laurent series inverting p over various coefficients respectively.

4 (ϕ, Γ)-modules

In this section, we prove a series of equivalences between the (1-)categories of Galois representations and (ϕ, Γ)-modules, which reduces Galois representations to some explicit objects that we can compute with.

Let R substitute for one of the letters E, A and B in this paragraph. Conceptually, for a finite extension K of $\mathbb{Q}_p$, the construction of R_K has and only has contained all the ramification information (by which we mean K_∞), so that H_K acts freely on R and $R^{H_K} = R_K$. Therefore, we can reduce Galois representations over R to the totally ramified part by taking the H_K-fixed points, which is controlled by Γ_K. On the other hand, the information of unramified part is determined by the action of Frobenius map ϕ.

4.1 Definition of (ϕ, Γ)-modules

In this subsection, suppose R is a commutative ring with an endomorphism σ.

Definition 4.1 (ϕ-module)**.** A ***ϕ-module over*** R is a R-module M together with a σ-semilinear endomorphism $\phi\colon M \to M$, i.e., ϕ is additive and $\phi(rm) = \sigma(r)\phi(m)$ for all $r \in R$ and $m \in M$.

Equivalently, a ϕ-module over R is a R-module M with a R-linear morphism $\Phi\colon M \to \sigma^* M$.

For simplicity, we will only denote a ϕ-module (M, Φ) by the underlying module M.

Definition 4.2 (Étale ϕ-module)**.** An ***étale ϕ-module over*** R is a ϕ-module M such that Φ is an isomorphism.

The following is an easy lemma. We omit the proof.

Lemma 4.3. *If D is a finite free ϕ-module of dimension n over R, then D is étale if and only if $\mathrm{Mat}(\phi) \in \mathrm{GL}_n(R)$.*

Suppose Γ is a group and R is equipped with an action of Γ, which commutes with σ.

Definition 4.4 ((ϕ, Γ)-module)**.** A (ϕ, Γ)-***module over*** R is a ϕ-module with a semilinear Γ-action commuting with ϕ.

Similarly, we will only denote a (ϕ, Γ)-module by its underlying module.

Suppose now that R and Γ are both equipped with Hausdorff and complete topology. In addition, suppose that R is a Noetherian flat R-algebra via the structure continuous map σ.

Definition 4.5 (Étale (ϕ, Γ)-module)**.** An ***étale*** (ϕ, Γ)-***module over*** R is a (ϕ, Γ)-module over R such that ϕ and the Γ-action is continuous, and it is étale as a ϕ-module.

Let $\mathrm{Mod}_R^{\text{ét}}(\phi, \Gamma)$ denote the abelian category of (ϕ, Γ)-modules over R (cf. [FO22, Proposition 3.19]).

Remark 4.6. *When we discuss étale (ϕ, Γ)-modules over E_K, we consider the topology given by val_E.*

When we discuss étale (ϕ, Γ)-modules over A_K and B_K, we consider the weak topology on them.

Notation. *For simplicity (and as done in many references, such as [Ber10]), we will assume that all ϕ-modules and (ϕ, Γ)-modules are finite free.*

4.2 (ϕ, Γ)-modules and p-adic Galois representations

Suppose G is a topological group, R is a topological commutative ring with a continuous G-action and M is a finite free R-module of dimension n with a continuous semilinear G-action. Pick a basis e for D. Then the map $G \to \mathrm{GL}_n(R)$ given by $g \mapsto \mathrm{Mat}_e(g)$ is an 1-cocycle in $C^1_{\mathrm{cts}}(G, \mathrm{GL}_n(R))$. If we choose another basis for D, the 1-cocycle will differ by an 1-coboundary. Furthermore, it gives us a (non-canonical) bijection of sets

$$\{\text{semilinear representations of } G \text{ of dimension } n\}/\text{isomorphisms} \cong H^1(G, \mathrm{GL}_n(R)).$$

By the above discussion and Theorem 3.9 and Theorem 3.4, we get the following.

Corollary 4.7. *Suppose K is a finite extension of $\mathbb{Q}_p$. Every semilinear representation of H_K of dimension n over A and E is (non-canonically) isomorphic to A^n and E^n respectively.*

In the rest of this section, suppose K is a finite extension of $\mathbb{Q}_p$.

Proposition 4.8. *Suppose V is a $\mathbb{F}_p$-representation of G_K of dimension n and $D(V) := (E \otimes_{\mathbb{F}_p} V)^{H_K}$. Then $D(V)$ is an étale (ϕ, Γ_K)-module over E_K of dimension n, $E \otimes_{E_K} D(V) \cong E \otimes_{\mathbb{F}_p} V$ in the category of (ϕ, G_K)-modules via the map $\lambda \otimes x \mapsto \lambda x$. In particular, $V \cong (E \otimes_{E_K} D(V))^{\phi=1}$ via the above isomorphism.*

Proof. By Corollary 4.7, $E \otimes_{\mathbb{F}_p} V \simeq E^n$ in the category of representations of H_K over E. Therefore, $D(V) \cong E_K^n$ in the category of E_K-modules. Since ϕ commutes with the H_K-action on E, $D(V)$ promotes to a ϕ-module over E_K. Since the remaining Γ_K-action on $D(V)$ acts trivially on E, it commutes with ϕ. Thus, $D(V)$ promotes to a (ϕ, Γ_K)-module over E_K.

Now we show that $D(V)$ is étale. Suppose $e = (e_i)$ is a $\mathbb{F}_p$-basis for V, $f = (f_i)$ is an E_K-basis for $D(V)$ and $f = eA$ for some $A \in \mathrm{GL}_n(E)$. Suppose $\phi(f) = fB$ for some $B \in \mathrm{Mat}_n(E_K)$. Then $e\phi(A) = \phi(f) = fB = eAB$. Thus, $B = A^{-1}\phi(A) \in \mathrm{GL}_n(E_K)$, which implies that $D(V)$ is étale.

Since $\dim(D(V)) = \dim(V)$, there is an E-basis of $E \otimes_{\mathbb{F}_p} V$ lives in $D(V)$. Thus, $E \otimes_{E_K} D(V) \cong E \otimes_{\mathbb{F}_p} V$ via the map $\lambda \otimes x \mapsto \lambda x$. This morphism commutes with ϕ and the G_K-action. Thus, this isomorphism promotes to an isomorphism in the category of (ϕ, Γ_K)-modules.

Since $\mathbb{F}_p = E^{\phi=1}$, $V \cong (E \otimes_{E_K} D(V))^{\phi=1}$. ∎

Actually, the functor D is an equivalence. To prove this, we need the following theorem.

Theorem 4.9 (cf. [Ber10, Theorem 8.6]). *If k is a separably closed field of characteristic p, and V is an étale ϕ-module over k, then V admits a basis fixed by ϕ and $1-\phi\colon V \to V$ is surjective.*

Proposition 4.10. *Suppose D is an étale (ϕ, Γ_K)-module over E_K of dimension n. Then $(E \otimes_{E_K} D)^{\phi=1}$ is a $\mathbb{F}_p$-representation of G_K of dimension n and $E \otimes_{\mathbb{F}_p} (E \otimes_{E_K} D)^{\phi=1} \cong E \otimes_{E_K} D$ in the category of (ϕ, G_K)-modules via the map $\lambda \otimes x \mapsto \lambda x$. In particular, $D \cong (E \otimes_{\mathbb{F}_p} (E \otimes_{E_K} D)^{\phi=1})^{H_K}$ via the above isomorphism.*

Proof. Since $E \cong \mathbb{F}_p((T))^{\mathrm{sep}}$ is separably closed of characteristic p, $E \otimes_{E_K} D$ admits a basis fixed by ϕ by Theorem 4.9. Therefore, $(E \otimes_{E_K} D)^{\phi=1}$ has dimension n.

The remaining proof is similar to the one of Proposition 4.8. □

Therefore, we have established the following equivalence of categories.

Theorem 4.11. *There is an equivalence of abelian categories $\mathrm{Rep}_{\mathbb{F}_p}(G_K) \cong \mathrm{Mod}_{E_K}^{\text{ét}}(\phi, \Gamma_K)$ given by $V \mapsto (E \otimes_{\mathbb{F}_p} V)^{H_K}$ and $D \mapsto (E \otimes_{E_K} D)^{\phi=1}$ for $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_K)$ and $D \in \mathrm{Mod}_{E_K}^{\text{ét}}(\phi, \Gamma_K)$.*

By Theorem 3.9, one can prove the following proposition mimicking the proof of Proposition 4.8.

Proposition 4.12. *Suppose V is a $\mathbb{Z}_p$-representation of G_K of dimension n and $D(V) := (A \otimes_{\mathbb{Z}_p} V)^{H_K}$. Then $D(V)$ is an étale (ϕ, Γ_K)-module over A_K of dimension n, $A \otimes_{A_K} D(V) \cong A \otimes_{\mathbb{Z}_p} V$ in the category of (ϕ, G_K)-modules via the map $\lambda \otimes x \mapsto \lambda x$. In particular, $V \cong (A \otimes_{A_K} D(V))^{\phi=1}$ via the above isomorphism.*

By successive approximation, we have the following corollary of Theorem 4.9.

Corollary 4.13. *If R is a commutative ring which is complete with respect to the p-adic topology, R/pR is a separably closed field of characteristic p, R is equipped with a Frobenius endomorphism ϕ lifting the Frobenius on R/pR, and V is an étale ϕ-module over R, then V admits a basis fixed by ϕ and $1-\phi\colon V \to V$ is surjective.*

Similarly, we have the following proposition for A and $\mathbb{Z}_p$ and the equivalence of categories.

Proposition 4.14. *Suppose D is an étale (ϕ, Γ_K)-module over A_K of dimension n. Then $(A \otimes_{A_K} D)^{\phi=1}$ is a $\mathbb{Z}_p$-representation of G_K of dimension n and $A \otimes_{\mathbb{Z}_p} (A \otimes_{A_K} D)^{\phi=1} \cong A \otimes_{A_K} D$ in the category of (ϕ, G_K)-modules via the map $\lambda \otimes x \mapsto \lambda x$. In particular, $D \cong (A \otimes_{\mathbb{Z}_p} (A \otimes_{A_K} D)^{\phi=1})^{H_K}$ via the above isomorphism.*

Theorem 4.15. *There is an equivalence of abelian categories $\mathrm{Rep}_{\mathbb{Z}_p}(G_K) \cong \mathrm{Mod}_{A_K}^{\text{ét}}(\phi, \Gamma_K)$ given by $V \mapsto (A \otimes_{\mathbb{Z}_p} V)^{H_K}$ and $D \mapsto (A \otimes_{A_K} D)^{\phi=1}$ for $V \in \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ and $D \in \mathrm{Mod}_{A_K}^{\text{ét}}(\phi, \Gamma_K)$.*

As said at the end of Section 3.2, there is no analog of Hilbert's theorem 90 for B. Hence, we can only derive the equivalence of categories from the results for A_K. To do this, we need to modify the definition for étale (ϕ, Γ_K)-modules over B_K as follows.

Definition 4.16 (Étale (ϕ, Γ_K)-modules over B_K)**.** An ***étale*** (ϕ, Γ_K)-***module over*** B_K is a (ϕ, Γ_K)-module D of dimension n over B_K such that there is a basis for D in which $\mathrm{Mat}(\phi) \in \mathrm{GL}_n(A_K)$.

Lemma 4.17. *Every continuous $\mathbb{Q}_p$-representation V of dimension n of G_K admits a $\mathbb{Z}_p$-lattice stable under G_K.*

Proof. Pick a basis for V. The basis spans a $\mathbb{Z}_p$-lattive $\mathcal{L}$ of V. Since $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$, $\mathrm{GL}_n(\mathbb{Z}_p) = \mathrm{GL}_n(\mathbb{Q}_p) \cap \mathrm{Mat}_n(\mathbb{Z}_p)$ is an open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Thus, the subgroup H of G_K consisting of elements g such that $g\mathcal{L} \subset \mathcal{L}$ is an open subgroup of G_K. Since G_K is compact, H is of finite index. Then $\sum_{g \in G} gT$ is a finite sum and is a stable $\mathbb{Z}_p$-lattice in V. □

Proposition 4.18. *Suppose V is a $\mathbb{Q}_p$-representation of G_K of dimension n and $D(V) := (B \otimes_{\mathbb{Q}_p} V)^{H_K}$. Then $D(V)$ is an étale (ϕ, Γ_K)-module over B_K of dimension n, $B \otimes_{B_K} D(V) \cong B \otimes_{\mathbb{Q}_p} V$ in the category of (ϕ, G_K)-modules via the map $\lambda \otimes x \mapsto \lambda x$. In particular, $V \cong (B \otimes_{B_K} D(V))^{\phi=1}$ via the above isomorphism.*

Proof. By the above lemma, pick a stable $\mathbb{Z}_p$-lattice $\mathcal{L}$ of B. By Theorem 3.9, $A \otimes_{\mathbb{Z}_p} \mathcal{L} \simeq A^n$ as an A-representation of H_K. Thus, $B \otimes_{\mathbb{Q}_p} V \simeq B^n$ as B-representations of H_K. The remaining proof is similar to Proposition 4.8.

It remains to show that $D(V)$ is étale. Note that

$$B_K \otimes_{A_K} D(\mathcal{L}) := B_K \otimes_{A_K} (A \otimes_{\mathbb{Z}_p} \mathcal{L})^{H_K} \cong (B_K \otimes_{A_K} A \otimes_{\mathbb{Z}_p} \mathcal{L})^{H_K} \cong (B \otimes_{\mathbb{Z}_p} \mathcal{L})^{H_K} \cong D(V)$$

in the category of (ϕ, Γ_K)-modules. Therefore, an A_K-basis for $D(\mathcal{L})$ induces a B_K-basis $D(V)$. Thus, $D(V)$ is étale. $\square$

Proposition 4.19. *Suppose D is an étale (ϕ, Γ_K)-module over B_K of dimension n. Then $(B \otimes_{B_K} D)^{\phi=1}$ is a $\mathbb{Q}_p$-representation of G_K of dimension n and $B \otimes_{\mathbb{Q}_p} (B \otimes_{B_K} D)^{\phi=1} \cong B \otimes_{B_K} D$ in the category of (ϕ, G_K)-modules via the map $\lambda \otimes x \mapsto \lambda x$. In particular, $D \cong (B \otimes_{\mathbb{Q}_p} (B \otimes_{B_K} D)^{\phi=1})^{H_K}$ via the above isomorphism.*

Proof. Since D is étale over B_K, there is a submodule D_0 of D such that $D \cong B_K \otimes_{A_K} D_0$ as ϕ, Γ_K-modules. Then Theorem 4.15 implies that $A \otimes_{A_K} D_0 \cong A \otimes_{\mathbb{Z}_p} (A \otimes_{A_K} D_0)^{\phi=1}$. Thus,

$$B \otimes_{B_K} D \cong B \otimes_{A_K} D_0 \cong B \otimes_{\mathbb{Z}_p} (A \otimes_{A_K} D_0)^{\phi=1} \cong B \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (A \otimes_{A_K} D_0)^{\phi=1})$$

in the category of (ϕ, Γ_K)-modules. Since $B^{\phi=1} = \mathbb{Q}_p$, $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (A \otimes_{A_K} D_0)^{\phi=1} \cong (B \otimes_{B_K} D)^{\phi=1}$. $\square$

Therefore, we have finally proved the following theorem.

Theorem 4.20. *There is an equivalence of abelian categories $\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \cong \mathrm{Mod}^{\text{ét}}_{B_K}(\phi, \Gamma_K)$ given by $V \mapsto (B \otimes_{\mathbb{Q}_p} V)^{H_K}$ and $D \mapsto (B \otimes_{B_K} D)^{\phi=1}$ for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $D \in \mathrm{Mod}^{\text{ét}}_{B_K}(\phi, \Gamma_K)$.*

5 Robba rings

In this section, we will just rush through the construction of the Robba ring and its properties without any proof. We recommend [Wan20] for detailed proofs.

5.1 Overconvergent elements

Recall that $\tilde{E} := \mathbb{C}_p^\flat$, $\tilde{A} := \mathbb{W}_P(\mathbb{C}_p^\flat)$ and for each $k > 0$, $w_k \colon \tilde{A} \to \mathbb{R} \cup \{+\infty\}$ is given by $w_k(x) := \inf_{i \leqslant k} \mathrm{val}_{\tilde{E}}(x_i)$ for any $x = \sum_{i>0} p^i [x_i] \in \mathbb{W}_P(\mathbb{C}_p^\flat)$.

For any $r > 0$, let

$$\tilde{A}^{\dagger, r} := \{x \in \tilde{A} \colon w_k(x) + k\frac{pr}{p-1} \geqslant 0 \text{ for all } k > 0 \text{ and } \lim_{k \to +\infty} \Big(w_k(x) + k\frac{pr}{p-1}\Big) = +\infty\}$$

Note that if $r_2 > r_1 > 0$, then $\tilde{A}^{\dagger, r_2} \supset \tilde{A}^{\dagger, r_1}$.

Lemma 5.1 (cf. [Wan20, Lemma 1.4]). *The set $\tilde{A}^{\dagger,r}$ is a subring of $\tilde{A}$ which is stable under $G_{\mathbb{Q}_p}$ and $\phi\colon \tilde{A}^{\dagger,r} \to \tilde{A}^{\dagger,pr}$ is a bijection.*

Let $\nu_r\colon \tilde{A}^{\dagger,r} \to \mathbb{R}_{\geqslant 0}$ given by $\nu_r(x) := \inf_{k>0}(w_k(x) + k\frac{pr}{p-1})$. The following lemma shows that this is a valuation on $\tilde{A}^{\dagger,r}$. It will make $\tilde{A}^{\dagger,r}$ into a complete valuation ring, whose topology is compatible with the action of $G_{\mathbb{Q}_p}$ and the map ϕ.

Lemma 5.2 (cf. [Wan20, Lemma 1.5]). *For any $r > 0$ and $x, y \in \tilde{A}^{\dagger,r}$,*

1. $\nu_r(x) = +\infty$ *if and only if* $x = 0$.
2. $\nu_r(x+y) \geqslant \inf(\nu_r(x), \nu_r(y))$.
3. $\nu_r(xy) = \nu_r(x) + \nu_r(y)$.
4. $\nu_{pr}(\phi(x)) = p\nu_r(x)$.
5. $\nu_r(px) = \nu_r(x) + \frac{pr}{r-1}$.
6. $\nu_r(\sigma(x)) = \nu_r(x)$ *for all* $\sigma \in G_{\mathbb{Q}_p}$.

Proposition 5.3 (cf. [Wan20, Proposition 1.7]). *The ring $\tilde{A}^{\dagger,r}$ is Hausdorff and complete with respect to the topology given by ν_r.*

Lemma 5.4. *For all $r > 0$, the action of $G_{\mathbb{Q}_p}$ on $\tilde{A}^{\dagger,r}$ is continuous and $\phi\colon \tilde{A}^{\dagger,r} \to \tilde{A}^{\dagger,pr}$ is a homeomorphism.*

As before, let $\tilde{B}^{\dagger,r} := \tilde{A}^{\dagger,r}[1/p]$. We can extend ν_r to $\tilde{B}^{\dagger,r}$ via the 5-th part of Lemma 5.2. Note that Lemma 5.2 also holds for $\tilde{B}^{\dagger,r}$. Furthermore, for each finite extension K of $\mathbb{Q}_p$, let $\tilde{B}^{\dagger,r}_K := (\tilde{B}^{\dagger,r})^{H_K}$ and $\tilde{A}^{\dagger,r}_K := (\tilde{A}^{\dagger,r})^{H_K}$. Let $\tilde{B}^{\dagger} := \cup_{r>0}\tilde{B}^{\dagger,r}$.

Remark 5.5. *The ring $\tilde{A}^{\dagger,r}$ is not the ring of integers in $\tilde{B}^{\dagger,r}$, but is the ring of integers in $\tilde{B}^{\dagger,r} \cap \tilde{A}$.*

Similarly, let $B^{\dagger,r} := \tilde{B}^{\dagger,r} \cap B$, $B^{\dagger,r}_K := (B^{\dagger,r})^{H_K}$, $B^{\dagger} := \cup_{r>0}B^{\dagger,r}$ and $B^{\dagger}_K := \cup_{r>0}B^{\dagger,r}_K$.

Proposition 5.6 (cf. [Wan20, Proposition 1.9]). *The ring $\tilde{B}^{\dagger}$ is a field. As a consequence, $\tilde{B}^{\dagger}_K, B^{\dagger}$ and $B^{\dagger}_K$ are fields.*

5.2 Robba rings

Lemma 5.7 (cf. [Ber10, Lemma 22.1])**.** *Suppose K is a finite extension of $\mathbb{Q}_p$. There exists $r(K) > 0$ and $\pi_K^\dagger \in A_K^{\dagger,r(K)}$, such that the image $\overline{\pi}_K$ of $\pi_K^\dagger$ in E_K is a uniformizer and $\pi_K^\dagger/[\overline{\pi}_K]_P$ is a unit in $A_K^{\dagger,r(K)}$.*

Suppose K is an extension of $\mathbb{Q}_p$ and $r > 0$. Let $\mathcal{A}_K^r$ be the ring of formal power series $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n$ with coefficients in $\mathcal{O}_K$, such that $\mathrm{val}_p(a_n) + nr \geqslant 0$ for all n and $\lim_{n\to-\infty}(\mathrm{val}_p(a_n) + nr) = +\infty$. For any $f \in \mathcal{A}_K^r$, define $\omega_r(f) = \inf_{n\in\mathbb{Z}}(\mathrm{val}_p(a_n) + nr)$. It can be easily shown that ω_r is a valuation on $\mathcal{A}_K^r$. Therefore, $\mathcal{A}_K^r$ is isomorphic to the ring of analytic functions with coefficients in $\mathcal{O}_K$ convergent on the annulus $\{0 < \mathrm{val}_p(T) \leqslant r\}$ and bounded by 1 with respect to the norm associated to ω_r. Let $\mathcal{B}_K^r := \mathcal{A}_K^r[1/p]$. We can also extend ω_r to $\mathcal{B}_K^r$. Then $\mathcal{B}_K^r$ is isomorphic to the ring of bounded analytic functions with coefficients in K convergent on the annulus $\{0 < \mathrm{val}_p(T) \leqslant r\}$.

Let $e_K := [K_\infty : (K_0)_\infty]$, which is the ramification index of $K_\infty/(\mathbb{Q}_p)_\infty$.

Theorem 5.8 (cf. [Wan20, Theorem 1.23])**.** *For all $r > r_K$,*

1. *there is an isomorphism of topological rings $\mathcal{A}_{K_0}^{\frac{1}{re_K}} \to A_K^{\dagger,r}$ given by $f \mapsto f(\pi_K^\dagger)$ such that $\frac{pr}{p-1}\omega_{\frac{1}{re_K}}(f) = \nu_r(f(\pi_K^\dagger))$, and*
2. *there is an isomorphism of topological rings $\mathcal{B}_{K_0}^{\frac{1}{re_K}} \to B_K^{\dagger,r}$ given by $f \mapsto f(\pi_K^\dagger)$ such that $\frac{pr}{p-1}\omega_{\frac{1}{re_K}}(f) = \nu_r(f(\pi_K^\dagger))$.*

6 Cherbonnier–Colmez's theorem

Recall that in Section 4.2, we proved the equivalences between p-adic Galois representations and (ϕ, Γ)-modules. In this subsection, we want to push the equivalence further to (ϕ, Γ)-modules over overconvergent elements, which is the Cherbonnier–Colmez's theorem. To do this, we need to introduce the technique by Colmez-Sen-Tate to overcome the absence of generalized Hilbert's theorem 90.

6.1 The Colmez–Sen–Tate conditions

Let K be a finite extension of $\mathbb{Q}_p$, $\tilde{\Omega}$ be a $\mathbb{Q}_p$-algebra and $\mathrm{val}_\Omega \colon \tilde{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be a map such that

1. $\mathrm{val}_\Omega(x) = +\infty$ if and only if $x = 0$.

2. $\mathrm{val}_\Omega(x + y) \geqslant \inf(\mathrm{val}_\Omega(x), \mathrm{val}_\Omega(y))$.

3. $\mathrm{val}_\Omega(xy) \geqslant \mathrm{val}_\Omega(x) + \mathrm{val}_\Omega(y)$.

4. $\mathrm{val}_\Omega(p) > 0$ and $\mathrm{val}_\Omega(px) = \mathrm{val}_\Omega(p) + \mathrm{val}_\Omega(x)$ if $x \in \tilde{\Omega}$.

Assume that $\tilde{\Omega}$ is complete with respect to the topology defined by val_Ω and $\tilde{\Omega}$ is equipped with a G_K-action such that val_Ω is G_K-invariant.

We say that $\tilde{\Omega}$ satisfies the Colmez–Sen–Tate conditions if there exists constants $c_1, c_2, c_3 \in \mathbb{R}_{\geqslant 0}$ such that the following three conditions hold.

(CST1) For every finite extension M/L of K, there exists $\alpha \in \tilde{\Omega}^{H_M}$ such that $\mathrm{val}_\Omega(\alpha) > -c_1$ and $\mathrm{Tr}_{M_\infty/L_\infty}(\alpha) = 1$.

(CST2) For every finite extension L of K, there exists $n(L) \in \mathbb{Z}_{>0}$ and an increasing sequence $\{\Omega_{L,n}\}_{n \geqslant n(L)}$ of closed sub-$\mathbb{Q}_p$-algebras of $\tilde{\Omega}^{H_L}$ along with maps $R_{L,n} \colon \tilde{\Omega}^{H_L} \to \Omega_{L,n}$ satisfying the following properties.

(a) If $x \in \tilde{\Omega}^{H_L}$, then $\mathrm{val}_\Omega(R_{L,n}(x)) \geqslant \mathrm{val}_\Omega(x) - c_2$ and $R_{L,n}(x) \to x$ as $n \to \infty$.

(b) If L_2/L_1 is finite, then $\Omega_{L_1,n} \subset \Omega_{L_2,n}$ and $R_{L_2,n}|_{\tilde{\Omega}^{H_{L_1}}} = R_{L_1,n}$.

(c) $R_{L,n}$ is $\Omega_{L,n}$-linear and is the identity on $\Omega_{L,n}$.

(d) If $g \in G_K$, then $g(\Omega_{L,n}) = \Omega_{g(L),n}$ and $g \circ R_{L,n} = R_{g(L),n} \circ g$.

Let $\Omega_{L,\infty} := \cup_{n \geqslant n(L)} \Omega_{L,n}$.

(CST3) For every finite extension L of K, there exists $m(L) \geqslant n(L)$ such that for all $\gamma \in \Gamma_L$ and $n \geqslant \sup(\mathrm{val}_p(\chi(\gamma) - 1), m(L))$, $1 - \gamma$ is invertible on $X_{L,n} := (1 - R_{L,n})(\tilde{\Omega}^{H_L})$ and $\mathrm{val}_\Omega((\gamma - 1)^{-1}x) \geqslant \mathrm{val}_\Omega(x) - c_3$ for all $x \in X_{L,n}$.

Example 6.1 (cf. [Ber10, §10 and §19]). *Let $\tilde{\Omega} := \mathbb{C}_p$ with p-adic valuation, $\Omega_{L,n} := L_n$ be the finite totally ramified extension over L constructed by Lubin-Tate and R_n is the Tate's normalized traces. Then $\tilde{\Omega}$ satisfies the CST conditions.*

The point of the CST conditions is that we can reduce $\mathrm{Rep}_{\tilde{\Omega}}(G_K)$ to $\mathrm{Rep}_{\Omega_{L,n}}(\mathrm{Gal}(L_\infty/K))$ for some finite extension L of K and $n \geqslant n(L)$. In particular, the reduction have two steps.

1. The condition CST1 helps us to reduce to $\mathrm{Rep}_{\tilde{\Omega}^{H_L}}(\mathrm{Gal}(L_\infty/K))$ (cf. [Ber10, Corollary 19.3]). This result is similar to the generalized Hilbert's theorem 90.
2. The conditions CST2 and CST3 together approximate $\mathrm{Rep}_{\tilde{\Omega}^{H_L}}(\mathrm{Gal}(L_\infty/K))$ through $\mathrm{Rep}_{\Omega_{L,n}}(\mathrm{Gal}(L_\infty/K))$ (cf. [Ber10, Corollary 19.5]).

To be precise, we have the following theorems.

Theorem 6.2 (cf. [Ber10, Theorem 19.1]). *If $\tilde{\Omega}$ satisfies the CST conditions, then*

$$\mathrm{colim}_L \mathrm{colim}_{n\geqslant n(L)} H^1(\mathrm{Gal}(L_\infty/K), \mathrm{GL}_d(\Omega_{L,n})) \cong H^1(G_K, \mathrm{GL}_d(\tilde{\Omega}))$$

where the isomorphism is induced by the inflation maps.

Theorem 6.3 (cf. [Ber10, Theorem 19.6 and Theorem 19.8]). *Suppose $W \in \mathrm{Rep}_{\tilde{\Omega}}(G_K)$ of dimension d. There exists a finite extension L of K and a finite free $\Omega_{L,\infty}$-submodule $W_{L,\infty} \subset W^{H_L}$ of dimension d such that $W_{L,\infty}$ is stable under $\mathrm{Gal}(L_\infty/K)$ and $W_{L,\infty} \otimes_{\Omega_{L,\infty}} \tilde{\Omega} \cong W$ in $\mathrm{Rep}_{\tilde{\Omega}}(G_K)$.*

Furthermore, $W_{L,\infty}$ is the greatest $\Omega_{L,\infty}$-sub-representation of $\mathrm{Gal}(L_\infty/K)$ of W^{H_L}.

Proposition 6.4 (cf. [Ber10, §24]). *Let $K := \mathbb{Q}_p$. There exists $r_K > 0$, such that for all $r > r_K$, $\tilde{\Omega} := \tilde{B}^{\dagger,r}$ with $\mathrm{val}_\Omega := \nu_r$ and $\Omega_{L,n} := \phi^{-n}(B_L^{\dagger,p^n r})$ satisfy the CST conditions with some maps $R_{L,n}$ defined in [Ber10, §24].*

6.2 Cherbonnier–Colmez's theorem

Theorem 6.5 (Cherbonnier–Colmez). *Suppose K is a finite extension of $\mathbb{Q}_p$. The functor $V \mapsto D^\dagger(V) := (B^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}$ induces an equivalence between $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $\mathrm{Mod}^{\text{et}}_{B_K^\dagger}(\phi, \Gamma_K)$, where a (ϕ, Γ_K)-modules over $B_K^\dagger$ is étale if it is after base-changing to B_K.*

By definition, $\mathrm{Mod}^{\text{et}}_{B_K^\dagger}(\phi, \Gamma_K) \cong \mathrm{Mod}^{\text{et}}_{B_K}(\phi, \Gamma_K)$ by base-changing. Thus, we remain to show that $D^\dagger$ is well-defined and $D(V)$ is naturally isomorphic to $B_K \otimes_{B_K^\dagger} D^\dagger(V)$, where $D(V) := (B \otimes_{\mathbb{Q}_p} V)^{H_K}$ as in Theorem 4.20.

The theorem is deduced from the following lemma and proposition. The idea is that firstly we use the CST-method to reduce to a $\mathrm{Gal}(L_\infty/K)$-submodule $D_L^{\dagger,r}$ of $D^\dagger(V)$ depending on the radius of convergence r. Since ϕ induces a homeomorphism $\tilde{B}^{\dagger,r} \to \tilde{B}^{\dagger,pr}$ for all $r > 0$ and the matrix of ϕ has only finite entries, we can raise the radius of convergence large enough to promote $D_L^{\dagger,r}$ to a (ϕ, Γ_K)-module. Finally, we extend the coefficient to get $D^\dagger(V)$.

Lemma 6.6. *For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of dimension d, there is a finite extension L of K and $s(V) \in \mathbb{R}_{>0}$ such that for all $s \geqslant s(V)$, $(\tilde{B}^{\dagger,s} \otimes_{\mathbb{Q}_p} V)^{H_L}$ has a free $B_L^{\dagger,s}$-submodule $D_L^{\dagger,s}$ of dimension d such that $D_L^{\dagger,s}$ is stable under G_K, $\tilde{B}^{\dagger,s} \otimes_{B_L^{\dagger,s}} D_L^{\dagger,s} \cong \tilde{B}^{\dagger,s} \otimes_{\mathbb{Q}_p} V$ via the map $\lambda \otimes x \mapsto \lambda x$ and $D_L^{\dagger} := B_L^{\dagger} \otimes_{B_L^{\dagger,s}} D_L^{\dagger,s} \hookrightarrow \tilde{B}^{\dagger} \otimes_{\mathbb{Q}_p} V$ is stable under ϕ.*

Proof. Fix $r > 0$ such that $(\tilde{B}^{\dagger,r}, \nu_r, \phi^{-n}(B_L^{\dagger,p^n r}))$ satisfies the CST conditions. By Theorem 6.2, there is a finite extension L of K, $n \in \mathbb{Z}_{>0}$ and a finite free $\phi^{-n}(B_L^{\dagger,p^n r})$-submodule $D_{L,n}^{\dagger,r}$ of $(\tilde{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V)^{H_L}$ such that $D_{L,n}^{\dagger,r}$ is of dimension d and stable under G_K and $\tilde{B}^{\dagger,r} \otimes_{\phi^{-n}(B_L^{\dagger,p^n r})} D_{L,n}^{\dagger,r} \cong \tilde{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V$.

We want the coefficient to be in $B_L^{\dagger,p^n r}$, but not in $\phi^{-n}(B_L^{\dagger,p^n r})$. Let $D_L^{\dagger,p^n r} := \phi^n(D_{L,n}^{\dagger,r})$ in $\tilde{B}^{\dagger} \otimes_{\mathbb{Q}_p} V$. Then $D_L^{\dagger,p^n r}$ is stable under G_K. Since ϕ is injective, $D_L^{\dagger,p^n r}$ is still finite free of dimension d. Moreover, we have $\tilde{B}^{\dagger,p^n r} \otimes_{B_L^{\dagger,p^n r}} D_L^{\dagger,p^n r} \cong \tilde{B}^{\dagger,p^n r} \otimes_{\mathbb{Q}_p} V$.

Now we have to deal with the action of ϕ. For any $t > 0$, let $B_{L,\infty}^{\dagger,t} := \cup_{n \geqslant n(L)} \phi^{-n}(B_L^{\dagger,p^n t})$. Note that $B_{L,\infty}^{\dagger,p^{n+1}r} \otimes_{B^{\dagger,p^n r}} D_L^{\dagger,p^{n+1}r}$ and $B_{L,\infty}^{\dagger,p^{n+1}r} \otimes_{\phi(B^{\dagger,p^n r})} \phi(D_L^{\dagger,p^{n+1}r})$ are both finite free $B_{L,\infty}^{\dagger,p^{n+1}r}$-submodules of $(\tilde{B}^{\dagger,p^{n+1}r} \otimes_{\mathbb{Q}_p} V)^{H_L}$ of dimension d and stable under G_K. Thus, by Theorem 6.3, there exists a finite free $B_{L,\infty}^{\dagger,p^{n+1}r}$-submodule $D_{L,\infty}^{\dagger,p^{n+1}r}$ of $(\tilde{B}^{\dagger,p^{n+1}r} \otimes_{\mathbb{Q}_p} V)^{H_L}$, such that the above two modules are contained in $D_{L,\infty}^{\dagger,p^{n+1}r}$. In particular, the matrix of ϕ under a basis of $D_L^{\dagger,p^n r}$ belongs to $\phi^{-m}(B_L^{\dagger,p^{m+n+1}r})$ for $m \in \mathbb{Z}_{>0}$ large enough.

We finish the proof by putting $s(V) := p^{m+n+1}r$. □

Proposition 6.7 (cf. [Wan20, Theorem 2.20(1)]). *For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of dimension d, let $D_L^{\dagger}$ be the finite free $(\phi, \mathrm{Gal}(L_\infty/K))$-module of dimension d over $B_L^{\dagger}$ in the above lemma. Then $B_L \otimes_{B_L^{\dagger}} D_L^{\dagger} \cong (B \otimes_{\mathbb{Q}_p} V)^{H_L} =: D_L(V)$ in $\mathrm{Mod}_{B_L}^{\text{ét}}(\phi, \Gamma_L)$.*

Moreover, $D^{\dagger}(V)$ is an étale (ϕ, Γ_K)-module over $B_K^{\dagger}$ of dimension d and $B_K \otimes_{B_K^{\dagger}} D^{\dagger}(V) \cong D(V)$ via the map $\lambda \otimes x \mapsto \lambda x$, which is natural.

Proof. Let $D_L^{\dagger}$ be the Moreover, $\tilde{B}^{\dagger} \otimes_{B_L^{\dagger}} D_L^{\dagger} \cong \tilde{B}^{\dagger} \otimes_{\mathbb{Q}_p} V$ via the map $\lambda \otimes x \mapsto \lambda x$. We want to compare both sides over rings without tilde.

Let $D_L := B_L \otimes_{B_L^{\dagger}} D_L^{\dagger}$. Then $\tilde{B} \otimes_{B_L} D_L \cong \tilde{B} \otimes_{\mathbb{Q}_p} V$. Let $\mathcal{L}$ be a lattice in V. Since $\tilde{B} = \tilde{A}[1/p]$ and B_L is a subfield of $\tilde{B}$, $D_L \cap \tilde{A} \otimes_{\mathbb{Z}_p} \mathcal{L}$ is an A_L-lattice in D_L. Thus, D_L is étale by a similar argument in Proposition 4.18. By Theorem 4.20, there is a $W \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ such that $\tilde{B} \otimes_{\mathbb{Q}_p} W \cong \tilde{B} \otimes_{B_L} D_L \cong \tilde{B} \otimes_{\mathbb{Q}_p} V$ as (ϕ, G_K)-modules over $\tilde{B}$. Since $\tilde{B}^{\phi=1} = \mathbb{Q}_p$, $W \cong V$ in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ by taking the ϕ-fixed points. Thus, $D_L \cong (B \otimes_{\mathbb{Q}_p} V)^{H_L}$ in $\mathrm{Mod}_{B_L}^{\text{ét}}(\phi, \Gamma_L)$.

Note that $D_L^\dagger$ admits compatible monomorphisms to both $\tilde{B}^\dagger \otimes_{\mathbb{Q}_p} V$ and $B \otimes_{\mathbb{Q}_p} V$. Therefore, there is a monomorphism $D_L^\dagger \hookrightarrow (B^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L}$ in $\mathrm{Mod}^{\text{ét}}_{B_L^\dagger}(\phi, \Gamma_L)$. Note that $\dim((B^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L}) \leqslant \dim(V) = \dim(D_L) = \dim(D_L^\dagger)$. We have that $D_L^\dagger \cong (B^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L}$. Similarly, we have $B_L \otimes_{B_L^\dagger} D_L^\dagger \cong D_L(V)$ in $\mathrm{Mod}^{\text{ét}}_{B_L}(\phi, \Gamma_L)$.

Finally, we use the Galois descent to reduce to the field K. Note that $B_K^\dagger = (B_L^\dagger)^{H_K/H_L}$. By Proposition 3.3, $H^1_{\text{cts}}(H_K/H_L, \mathrm{GL}_d(B_L^\dagger)) \cong 0$. Therefore, $D^\dagger(V) \cong \left((B^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L}\right)^{H_K/H_L}$ is of dimension d, and $D^\dagger(V)$ is étale since D_L is étale. Thus, $B_L^\dagger \otimes_{B_K^\dagger} D^\dagger(V) \cong D_L^\dagger$. Hence,

$$B \otimes_{B_K^\dagger} D^\dagger(V) \cong B \otimes_{B_L^\dagger} D_L^\dagger \cong B \otimes_{B_L} D_L(V) \cong B \otimes_{\mathbb{Q}_p} V.$$

By taking H_K-fixed points on both sides, we get $B_K \otimes_{B_K^\dagger} D^\dagger(V) \cong D(V)$ in $\mathrm{Mod}^{\text{ét}}_{B_K}(\phi, \Gamma_L)$.

□

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